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Hanner-type inequality and optimal 2-uniform convexity and smoothness inequalities

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Hanner は L_p の modulus of convexity を決定する際, Hanner 不等式を用いた [2]. ここではそれを包括する重み γ 付きの Hanner 型不等式とその多元版を導入し，双対性, optimal 2-uniform convexity (smoothness) 不等式との関係を考察する. 特に L_p に対しては最良の γ を決定する.

Theorem 1 (Hanner inequality).

(H1) $$\|x+y\|^p+\|x-y\|^p \geq \Big|\|x\|+\|y\|\Big|^p+\Big|\|x\|-\|y\|\Big|^p \qquad \forall x,y\in L_p \quad (1<p\leq 2)$$

(H2) $$\|x+y\|^p+\|x-y\|^p \leq \Big|\|x\|+\|y\|\Big|^p+\Big|\|x\|-\|y\|\Big|^p \qquad \forall x,y\in L_p \quad (2\leq p<\infty)$$

この Hanner の不等式は L_p 空間の他, p–Schatten class operator のなす空間 C_p では $1\leq p\leq 4/3$ のとき (H1), $p\geq 4$ のとき (H2) が成立する. なお $p=1$ の場合，(H1) は三角不等式となりすべてのノルム空間で成立する. ここでは (H1), (H2) に重みをつけて一般化した Hanner 型不等式を導入し, この不等式を満たすようなバナッハ空間を考察する.

Theorem 2 (Hanner type inequality). Let X be a Banach space and let $1<p,s,t<\infty$.

(i) The inequality

($\mathrm{H1}_\gamma$) $$\|x+y\|^p+\|x-y\|^p \geq \Big|\|x\|+\|\gamma y\|\Big|^p+\Big|\|x\|-\|\gamma y\|\Big|^p$$

holds in X with some $\gamma>0$ if and only if the inequality

(HT1) $$\left(\frac{\|x+y\|^s+\|x-y\|^s}{2}\right)^{1/s} \geq \left(\frac{\Big|\|x\|+\|\gamma y\|\Big|^t+\Big|\|x\|-\|\gamma y\|\Big|^t}{2}\right)^{1/t}.$$

holds in X with some $\gamma > 0$.

(ii) The inequality

$$(\mathrm{H2}_\gamma) \qquad \|x+y\|^p + \|x-y\|^p \le \Big|\|x\| + \|\gamma y\|\Big|^p + \Big|\|x\| - \|\gamma y\|\Big|^p$$

holds in X with some $\gamma > 0$ if and only if the inequality

$$(\mathrm{HT2}) \qquad \left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \le \left(\frac{\Big|\|x\| + \|\gamma y\|\Big|^t + \Big|\|x\| - \|\gamma y\|\Big|^t}{2}\right)^{1/t}$$

holds in X with some $\gamma > 0$.

$(\mathrm{H1}_\gamma)$, $(\mathrm{H2}_\gamma)$ は Hanner 不等式に重み γ を付加したもので, (HT1), (HT2) はそれぞれを一般化した不等式である. (HT1), (HT2) の不等式の右辺は γ について単調増加関数であり (HT1) では $0 < \gamma \le 1$, (HT2) では $1 \le \gamma < \infty$ となる. 我々は Hanner 型不等式 (HT1) または (HT2) を満たすバナッハ空間 X と最良の γ を考察する.

Hanner 型不等式の成立するバナッハ空間の双対空間において, 逆向きの Hanner 型不等式が成立する.

Theorem 3 (Duality). Let X be a Banach space and X^* the dual space of X. Let $1 < s, t < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1$ and $\gamma > 0$. Then the following are equivalent.

(i) The inequality

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \ge \left(\frac{\Big|\|x\| + \|\gamma y\|\Big|^t + \Big|\|x\| - \|\gamma y\|\Big|^t}{2}\right)^{1/t}$$

holds in X.

(ii) The inequality

$$\left(\frac{\|x^*+y^*\|^{s'} + \|x^*-y^*\|^{s'}}{2}\right)^{1/s'} \le \left(\frac{\Big|\|x^*\| + \|\gamma^{-1} y^*\|\Big|^{t'} + \Big|\|x^*\| - \|\gamma^{-1} y^*\|\Big|^{t'}}{2}\right)^{1/t'}$$

holds in X^*.

これによりオリジナルな Hanner 不等式の証明は (H1) のみの証明で十分であることがわかる.

バナッハ空間Xが*q-uniformly convex* $(2 \leq q < \infty)$であるとは, 正数Cが存在して, 任意の$\epsilon > 0$に対し$\delta_X(\epsilon) \geq C\epsilon^q$が成立することである. また$X$が*p-uniformly smooth* $(1 < p \leq 2)$であるとは, 正数Kが存在して, 任意の$\tau > 0$に対し$\rho_X(\tau) \leq K\tau^p$が成立することである. ここで$\delta_X(\epsilon)$はmodulus of convexity; $\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\right\}$, $\rho_X(\tau)$はmodulus of smoothness; $\rho_X(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x - \tau y\|) -1 : \|x\| = \|y\| = 1\right\}$ である.

Lemma 4 (2-uniform convexity inequalities; [6]**).** Let X be a Banach space and let $1 < s < \infty$. Then the following are equivalent.

(i) X is 2-uniformly convex.

(ii) There exists $C > 0$ for which

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \geq \left(\|x\|^2 + \|Cy\|^2\right)^{1/2}$$

holds in X.

(iii) There exists $C > 0$ such that for any $n \in \mathbb{N}$

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|^s\right)^{1/s} \geq \left(\|x_1\|^2 + \sum_{j=2}^{n} \|Cx_j\|^2\right)^{1/2}$$

holds in X, where $\{\epsilon_j\}$ is a Rademacher sequence and $\mathbf{E}$ denotes the expectation.

In the case of $1 < s \leq 2$, one can take the same constant C in (ii) and (iii).

Lemma 5 (2-uniform smoothness inequalities; [6]**).** Let X be a Banach space and let $1 < s < \infty$. Then the following are equivalent.

(i) X is 2-uniformly smooth.

(ii) There exists $K > 0$ for which

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq \left(\|x\|^2 + \|Ky\|^2\right)^{1/2}$$

holds in X.

(iii) There exists $K > 0$ such that for any $n \in \mathbb{N}$

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|^s\right)^{1/s} \leq \left(\|x_1\|^2 + \sum_{j=2}^{n} \|Kx_j\|^2\right)^{1/2}$$

holds in X.

In the case of $s \geq 2$, one can take the same constant K in (ii) and (iii).

Theorem 6. Let X be a Banach space and let $1 < s, t < \infty$. Then the following are equivalent.

(i) X is 2-uniformly convex.

(ii) There exists some $\gamma > 0$ for which

$$\text{(HT1)} \qquad \left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \geq \left(\frac{\Big|\|x\| + \|\gamma y\|\Big|^t + \Big|\|x\| - \|\gamma y\|\Big|^t}{2}\right)^{1/t}$$

holds in X.

(iii) There exists some $\gamma > 0$ such that for any $n \in \mathbb{N}$

$$\text{(mHT1)} \qquad \left(\mathbf{E}\left\|\sum_{j=1}^n \epsilon_j x_j\right\|^s\right)^{1/s} \geq \left(\mathbf{E}\left|\epsilon_1\|x_1\| + \gamma\sum_{j=2}^n \epsilon_j\|x_j\|\right|^t\right)^{1/t}$$

holds in X.

Theorem 7. Let X be a Banach space and let $1 < s, t < \infty$. Then the following are equivalent.

(i) X is 2-uniformly smooth.

(ii) There exists some $\gamma > 0$ for which

$$\text{(HT2)} \qquad \left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq \left(\frac{\Big|\|x\| + \|\gamma y\|\Big|^t + \Big|\|x\| - \|\gamma y\|\Big|^t}{2}\right)^{1/t}$$

holds in X.

(iii) There exists some $\gamma > 0$ such that for any $n \in \mathbb{N}$

$$\text{(mHT2)} \qquad \left(\mathbf{E}\left\|\sum_{j=1}^n \epsilon_j x_j\right\|^s\right)^{1/s} \leq \left(\mathbf{E}\left|\epsilon_1\|x_1\| + \gamma\sum_{j=2}^n \epsilon_j\|x_j\|\right|^t\right)^{1/t}$$

holds in X.

(mHT1), (mHT2) は Hanner 型不等式の多元版である. 定数 γ は (mHT1) では $0 < \gamma \leq 1$, (mHT2) では $1 \leq \gamma < \infty$ で, これら不等式の右辺も 2 元の場合と同様 γ に関して単調増加関数である. したがって (mHT1), (mHT2) をみたす最良定数 γ を次のように定義する.

$$\gamma_{(s,t)}(X) := \sup\{\gamma > 0 : X \text{ で (mHT1) が成立する}\},$$

$$\gamma^{(s,t)}(X) := \inf\{\gamma > 0 : X \text{ で (mHT2) が成立する}\}.$$

2元についてのγの最良定数を

$$\gamma_{(s,t;2)}(X) := \sup\{\gamma > 0 : X \text{ で (HT1) が成立する} \},$$

$$\gamma^{(s,t;2)}(X) := \inf\{\gamma > 0 : X \text{ で (HT2) が成立する} \}$$

とすれば$\gamma_{(s,t;2)}(X) \geq \gamma_{(s,t)}(X)$, $\gamma^{(s,t;2)}(X) \leq \gamma^{(s,t)}(X)$は明らかである. 一般のバナッハ空間における最良定数の算出は容易ではないがL_p空間については次の結果を得る.

Theorem 8.
(i) $\gamma_{(s,t)}(L_p) = \min\left\{1, \sqrt{(p-1)/(t-1)}, \sqrt{(s-1)/(t-1)}\right\} \quad (1 < p \leq 2).$

(ii) $\gamma^{(s,t)}(L_p) = \max\left\{1, \sqrt{(p-1)/(t-1)}, \sqrt{(s-1)/(t-1)}\right\} \quad (2 \leq p < \infty).$

Further

$$\gamma_{(s,t)}(L_p) = \gamma_{(s,t;2)}(L_p) \quad (1 < p \leq 2),$$

$$\gamma^{(s,t)}(L_p) = \gamma^{(s,t;2)}(L_p) \quad (2 \leq p < \infty).$$

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